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On the Mittag-Leffler Property

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Abstract: Let C be a category with strong monomorphic strong coimages, that is, every morphism f of C factors as $f = u \circ g$ so that g is a strong epimorphism and u is a strong monomorphism and this factorization is universal. We define the notion of strong Mittag-Leffler property in pro-C. We show that if $f : X \to Y$ is a level morphism in pro-C such that $p(Y)_\alpha^\beta$ is a strong epimorphism for all $\beta > \alpha$, then X has the strong Mittag-Leffler property provided f is an isomorphism. Also, if $f : X \to Y$ is a strong epimorphism of pro-C and X has the strong Mittag-Leffler property, we show that Y has the strong Mittag-Leffler property. Moreover, we show that this property is invariant of isomorphisms of pro-C.

Keywords: Pro-categories, strong Mittag-Leffler property, categories with strong monomorphic strong coimages.
MSC: Primary 16B50.

1. INTRODUCTION

In [1], J. Dydak and F. R. Ruiz del Portal generalized the notion of Mittag-Leffler property to arbitrary balanced categories with epimorphic images. They obtained several results.

In [2], the author defined the notion of categories with strong monomorphic strong coimages. C is a category with strong monomorphic strong coimages if every morphism f of C factors as $f = u \circ g$ so that g is a strong epimorphism and u is a strong monomorphism and this factorization is universal among such factorization. In this paper, we define the notion of strong Mittag-Leffler property in pro-C. We show that if $f : X \to Y$ is a level morphism in pro-C such that $p(Y)_\alpha^\beta$ is a strong epimorphism for all $\beta > \alpha$, then X has the strong Mittag-Leffler property provided f is an isomorphism (Theorem 3.2). Also, if $f : X \to Y$ is a strong epimorphism of pro-C and X has the strong Mittag-Leffler property, we show that Y has the strong Mittag-Leffler property (Corollary 3.6). Moreover, we show that this property is invariant of isomorphisms of pro-C (Corollary 3.5).

2. PRELIMINARIES

First we recall some basic facts about pro-categories. The main reference is [3] and for more details see [4].

Let C be an arbitrary category. Loosely speaking, the pro-category pro-C of C is the universal category with inverse limits containing C as a full subcategory. An object of pro-C is an inverse system in C, denoted by $X = \left(X_\alpha, p_\alpha^\beta, A\right)$, consisting of a directed set A, called the *index set*, of C objects X_α for each $\alpha \in A$, called the *terms* of X and of C morphisms $p_\alpha^\beta : X_\beta \to X_\alpha$ for each related pair $\alpha < \beta$, called the *bonding morphisms* of X. A morphism of two objects $f : X = \left(X_\alpha, p_\alpha^\beta, A\right) \to Y = \left(Y_{\alpha'}, p_{\alpha'}^{\beta'}, A'\right)$ consists of a function $\varphi : A' \to A$ and of morphisms $f_{\alpha'} : X_{\varphi(\alpha')} \to Y_{\alpha'}$ in C one for each $\alpha' \in A'$ such that whenever $\alpha' < \beta'$, then there is $\gamma \in A$, $\gamma > \varphi(\alpha'), \varphi(\beta')$ for which $f_{\alpha'} p_{\phi(\alpha')}^{\gamma} = p_{\alpha'}^{\beta'} f_{\beta'} p_{\phi(\beta')}^{\gamma}$. From now onward, the index set A of an object X of pro-C will be denoted by $I(X)$ and the bonding morphisms by $p(X)_\alpha^\beta$ for each $\alpha < \beta$.

If P is an object of C and X is an object of pro-C, then a morphism $f : X \to P$ in pro-C is the direct limit of Mor $\left(X_\alpha, P\right)$, $\alpha \in I(X)$ and so f can be represented by $g : X_\alpha \to P$. Note that the morphism from X to X_α represented by the identity $X_\alpha \to X_\alpha$ is called the *projection morphism* and denoted by $p(X)_\alpha$.

If X and Y are two objects in pro-C with identical index sets, then a morphism $f : X \to Y$ is called a *level morphism* if for each $\alpha < \beta$, the following diagram commutes.

$$\begin{array}{ccc} X_\beta & \xrightarrow{f_\beta} & Y_\beta \\ {\scriptstyle p(X)_\alpha^\beta}\downarrow & & \downarrow{\scriptstyle p(Y)_\alpha^\beta} \\ X_\alpha & \xrightarrow{f_\alpha} & Y_\alpha \end{array}$$

Theorem 2.1. For any morphism $f : X \to Y$ of pro-C there exists a level morphism $f' = X' \to Y'$ and isomorphisms $i : X \to X'$, $j : Y' \to Y$ such that $f = j \circ f' \circ i$ and $I(X')$ is a

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